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# Investigation of anti-plane shear behavior of a Griffith crack in a piezoelectric material by using the non-local theory

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**Abstract.** In this paper, the behavior of a Griffith crack in a piezoelectric material under anti-plane shear loading is investigated by using the non-local theory for impermeable crack surface conditions. By using the Fourier transform, the problem can be solved with two pairs of dual integral equations. These equations are solved using Schmidt method. Numerical examples are provided. Contrary to the previous results, it is found that no stress and electric displacement singularity is present at the crack tip.

**Key words:** Crack, dual integral equations, Fourier transform, non-local theory, piezoelectric materials, Smidt theory.

## 1. Introduction

It is well known that piezoelectric materials produce an electric field when deformed, and undergo deformation when subjected to an electric field. The coupling nature of piezoelectric materials has attracted wide applications in electric-mechanical and electric devices, such as electric-mechanical actuators, sensors and structures. When subjected to mechanical and electrical loads in service, these piezoelectric materials can fail prematurely due to their brittleness and presence of defects or flaws produced during their manufacturing process. Therefore, it is important to study the electro-elastic interaction and fracture behaviors of piezoelectric materials.

Many studies have been made on the electro-elastic fracture mechanics based on the modeling and analyzing of one crack in the piezoelectric materials (see, for example, Deeg, 1980; Pak, 1990, 1992; Sosa, 1992; Suo et al., 1992; Park and Sun, 1995a,b; Zhang and Tong, 1996; Zhang et al., 1998; Gao et al., 1997; Wang, 1992). The problem of the interacting fields among multiple cracks in a piezoelectric materials has been studied by Han (Han et al., 1999). In Han's paper, the crack is treated as a continuous distributed dislocations with the density function to be determined according to the conditions of external loads and crack surface. Most recently, Chen and Karihaloo (1999) considered an infinite piezoelectric ceramic with impermeable crack-face boundary condition under arbitrary electro-mechanical impact. Sosa and Hhutoryansky (1999) investigated the response of piezoelectric bodies disturbed by internal electric sources. The impermeable boundary condition on the crack surface was widely used in the works (Pak, 1990, 1992; Suo et al., 1992; Suo, 1993; Park and Sun, 1995a,b; Chen and Karihaloo, 1999). However, these solutions contain stress and electric displacement singularity. This is not reasonable according to the physical nature. For overcoming the stress singularity in the classical elastic theory, Eringen (Eringen et al., 1977a, 1978, 1979) used the non-local theory to discuss the state of stress near the tip of a sharp line crack in an elastic plate subject to uniform tension, shear and anti-plane shear. These solutions did not contain any stress singularity, thus resolving a fundamental problem that persisted over many years. This enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way.

In the present paper, the behavior of a Griffith crack subjects to anti-plane shear in piezoelectric materials is investigated by using the non-local theory for impermeable crack surface conditions. The traditional concept of linear elastic fracture mechanics and the non-local theory are extended to include the piezoelectric effects. Fourier transform is applied and a mixed boundary value problem is reduced to two pairs of dual integral equations. In solving the dual integral equations, the crack surface displacement and electric potential are expanded in a series using Jacobi polynomials and Schmidt method (Morse et al., 1958) is used. This process is quite different from that adopted in references (Han et al., 1999; Deeg, 1980; Pak, 1990, 1992; Sosa, 1992; Suo et al., 1992; Par and Sun, 1995a,b; Zhang and Tong, 1996; Zhang et al., 1998; Gao et al., 1997; Wang, 1992; Eringen et al., 1977a, 1978, 1979). As expected, the solution in this paper does not contain the stress and electric displacement singularity at the crack tip, thus clearly indicating the physical nature of the problem, namely, in the vicinity of the geometrical discontinuities in the body, the non-local intermolecular forces are dominant. For such problems, therefore, one must resort to theories incorporating non-local effects, at least in the neighborhood of the discontinuities.

## 2. Basic equations of non-local piezoelectric materials

For the anti-plane shear problem, the basic equations of linear, homogeneous, isotropic, non-local piezoelectric materials, with vanishing body force are (see e.g. Eringen, 1979; Shindo, Narita and Tanakd, 1996)

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0, \tag{1}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0, \tag{2}$$

$$\tau_{kz}(X) = \int_V [c'_{44}(|X' - X|)w_{,k}(X') + e'_{15}(|X' - X|)\phi_{,k}(X')]\,\mathrm{d}V(X'), \tag{3}$$

$$D_k(X) = \int_V [e'_{15}(|X' - X|)w_{,k}(X') - \varepsilon'_{11}(|X' - X|)\phi_{,k}(X')]\,\mathrm{d}V(X'), \tag{4}$$

where the only difference from classical elastic theory and the pizoelectric theory is in the stress and the electric displacement constitutive Equations (3) and (4) in which the stress $\tau_{zk}(X)$ and the electric displacement $D_k(X)$ at a point $X$ depends on $w_{,k}(X)$ and $\phi_{,k}(X)$, at all points of the body. $w$ and $\phi$ are the mechanical displacement and electric potential. For homogeneous and isotropic piezoelectric materials there exist only three material parameters, $c'_{44}(|X' - X|)$, $e'_{15}(|X' - X|)$ and $\varepsilon'_{11}(|X' - X|)$ which are functions of the distance $|X' - X|$. The integrals in (3) and (4) are over the volume $V$ of the body enclosed within a surface $\partial V$.

As discussion in the papers (see, e.g., Eringen, 1974, 1977b), it can be assumed in the form of $c'_{44}(|X' - X|)$, $e'_{15}(|X' - X|)$ and $\varepsilon'_{11}(X' - X|)$ for which the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves the following has been found to be very useful

$$(c'_{44}, e'_{15}, \varepsilon'_{11}) = (c_{44}, e_{15}, \varepsilon_{11})\alpha(|X' - X|), \tag{5}$$

$$\alpha(|X' - X|) = \alpha_0 \exp[-(\beta/a)^2(X' - X)(X' - X)], \tag{6}$$

where $\beta$ is a constant, $a$ is the lattice parameter. $c_{44}, e_{15}, \varepsilon_{11}$ are the shear modulus, piezoelectric coefficient and dielectric parameter, respectively. $\alpha_0$ is determined by the normalization

$$\int_V \alpha(|X' - X|)\, \mathrm{d}V(X') = 1. \tag{7}$$

In the present work, the non-local elastic moduli was given by (5) and (6). Substituting (6) into (7), it can be obtained, in two-dimensional space,

$$\alpha_0 = \frac{1}{\pi}(\beta/a)^2. \tag{8}$$

Substitution of Equations (5) and (6) into Equations (3) and (4) yields

$$\tau_{kz}(X) = \int_V \alpha(|X' - X|)\sigma_{kz}(X')\, \mathrm{d}V(X'), \tag{9}$$

$$D_k(X) = \int_V \alpha(|X' - X|)D_k^c(X')\, \mathrm{d}V(X'), \tag{10}$$

where

$$\sigma_{kz} = c_{44}w_{,k} + e_{15}\phi_{,k}, \tag{11}$$

$$D_k^c = e_{15}w_{,k} - \varepsilon_{11}\phi_{,k}. \tag{12}$$

The expressions (11) and (12) are the classical constitutive equations.

## 3. The crack model

Consider an infinite piezoelectric body containing a Griffith impermeable crack of length $2l$ along the $x$-axis. The piezoelectric boundary-value problem for anti-plane shear is considerably simplified if we consider only the out-of-plane displacement and the in-plane elastic fields. The plate is subjected to a constant stress $\tau_{yz} = \tau_0$, and a constant electric displacement $D_y = D_0$ along the surface of the cracks, see Figure 1. As discussion in Narita's (Narita and Shindo, 1998), Shindo's (Shindo, 1996), Yu's (Yu, 1998) and Eringen's (Eringen, 1979) references, the boundary conditions of the present problem are:

$$\tau_{yz}(x, 0) = \tau_0, \quad |x| \le l, \tag{13}$$

$$D_y(x, 0) = D_0, \quad |x| \le l, \tag{14}$$

$$w(x, 0) = \phi(x, 0) = 0, \quad |x| > l, \tag{15}$$

$$w(x, y) = \phi(x, y) = 0 \quad \text{for} \quad (x^2 + y^2)^{1/2} \to \infty. \tag{16}$$

Substituting Equations (9) and (10) into Equations (1) and (2), respectively, using Green–Gauss theorem, it can be obtained (see, e.g., Eringen, 1979):

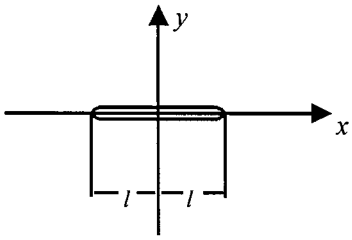


*Figure 1.* Crack in a piezoelectric material body under anti-plane shear.

$$\iint_V \alpha(|x'-x|,|y'-y|)[c_{44}\nabla^2 w(x',y') + e_{15}\nabla^2\phi(x',y')]\,\mathrm{d}x'\,\mathrm{d}y'$$

$$-\int_{-l}^{l}\alpha(|x'-x|,0)[\sigma_{yz}(x',0)]\,\mathrm{d}x' = 0, \tag{17}$$

$$\iint_V \alpha(|x'-x|,|y'-y|)[e_{15}\nabla^2 w(x',y') - \varepsilon_{11}\nabla^2\phi(x',y')]\,\mathrm{d}x'\,\mathrm{d}y'$$

$$-\int_{-l}^{l}\alpha(|x'-x|,0)[D_y^c(x',0)]\,\mathrm{d}x' = 0, \tag{18}$$

where the boldface bracket indicates a jump at the crack line. $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two dimensional Laplace operator. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0 \le x \le \infty$, $0 \le y \le \infty$ only. Under the applied anti-plane shear load on the unopened surfaces of the crack, the displacement field and the electric displacement possess the following symmetry regulations

$$w(x,-y) = -w(x,y), \qquad \phi(x,-y) = -\phi(x,y). \tag{19}$$

Using Equation (19), we find that

$$[\sigma_{yz}(x,0)] = 0, \tag{20}$$

$$[D_y^c(x,0)] = 0. \tag{21}$$

Hence, the line integrals in (17) and (18) vanish. By taking the Fourier transform of (17) and (18) with respect to $x'$, it can be shown that the general solutions of (17) and (18) are identical to that of

$$c_{44}\left[\frac{\mathrm{d}^2\overline{w}(s,y)}{\mathrm{d}y^2} - s^2\overline{w}(s,y)\right] + e_{15}\left[\frac{\mathrm{d}^2\overline{\phi}(s,y)}{\mathrm{d}y^2} - s^2\overline{\phi}(s,y)\right] = 0, \tag{22}$$

$$e_{15}\left[\frac{\mathrm{d}^2\overline{w}(s,y)}{\mathrm{d}y^2} - s^2\overline{w}(s,y)\right] - \varepsilon_{11}\left[\frac{\mathrm{d}^2\overline{\phi}(s,y)}{\mathrm{d}y^2} - s^2\overline{\phi}(s,y)\right] = 0, \tag{23}$$

almost everywhere. Here a superposed bar indicates the Fourier transform, e.g.

$$\overline{f}(s, y) = \int_0^\infty f(x, y)\exp(isx)\,\mathrm{d}x.$$

The general solutions of Equations (22) and (23) ($y \geq 0$) satisfying (16) are, respectively:

$$w(x, y) = \frac{2}{\pi}\int_0^\infty A(s)e^{-sy}\cos(xs)\,\mathrm{d}s,$$
$$\phi(x, y) - \frac{e_{15}}{\varepsilon_{11}}w(x, y) = \frac{2}{\pi}\int_0^\infty B(s)e^{-sy}\cos(xs)\,\mathrm{d}s, \tag{24}$$

where $A(s)$, $B(s)$ are to be determined from the boundary conditions.

The stress field and the electric displacement, according to (9) and (10), are given by, respectively

$$\tau_{yz}(x, y) = \frac{2}{\pi}\int_0^\infty [-\mu s A(s) - e_{15}sB(s)]\,\mathrm{d}s\int_0^\infty \mathrm{d}y'\int_{-\infty}^\infty [\alpha(|x'-x|, |y'-y|)$$
$$+\alpha(|x'-x|, |y'+y|)]e^{-sy'}\cos(sx')\,\mathrm{d}x', \tag{25}$$

$$D_y(x, y) = \frac{2}{\pi}\int_0^\infty \varepsilon_{11}sB(s)\,\mathrm{d}s\int_0^\infty \mathrm{d}y'\int_{-\infty}^\infty [\alpha(|x'-x|, |y'-y|)$$
$$+\alpha(|x'-x|, |y'+y|)]e^{-sy'}\cos(sx')\,\mathrm{d}x'. \tag{26}$$

Substituting for $\alpha$ from (6), according to the reference (see, e.g., Eringen, 1979) and the boundary conditions (13)–(15), it can be obtained

$$\frac{2}{\pi}\int_0^\infty sA(s)\,\mathrm{erfc}(\varepsilon s)\cos(sx)\,\mathrm{d}s = -\frac{1}{\mu}\left(\tau_0 + \frac{e_{15}D_0}{\varepsilon_{11}}\right), \quad |x| \leq l, \tag{27}$$

$$\frac{2}{\pi}\int_0^\infty A(s)\cos(sx)\,\mathrm{d}s = 0, \quad |x| > l \tag{28}$$

and

$$\frac{2}{\pi}\int_0^\infty sB(s)\,\mathrm{erfc}(\varepsilon s)\cos(sx)\,\mathrm{d}s = \frac{D_0}{\varepsilon_{11}}, \quad |x| \leq l, \tag{29}$$

$$\frac{2}{\pi}\int_0^\infty B(s)\cos(sx)\,\mathrm{d}s = 0, \quad |x| > l, \tag{30}$$

where

$$\varepsilon = \frac{a}{2\beta}, \quad \mathrm{erfc}(z) = 1 - \Phi(z), \quad \Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)\,\mathrm{d}t, \quad \mu = c_{44} + \frac{e_{15}^2}{\varepsilon_{11}}.$$

Since the only difference between the classical and the non-local equations is in the introduction of the function $\mathrm{erfc}(\varepsilon s)$, it is logical to utilize the classical solution to convert the system (27)–(30) to an integral equation of the second kind which is generally better behaved. For $a = 0$, then $\mathrm{erfc}(\varepsilon s) = 1$ and Equations (27)–(30) reduce to the dual integral equations for the same problem in classical piezoelectric materials. To determine the unknown function $A(s)$ and $B(s)$, the dual-integral equations (27)–(30) must be solved.

## 4. Solution of the dual integral equation

The dual integral equations (27)–(30) can not be transformed into the second Fredholm integral equation (Eringen, 1979), because the kernel of the second kind Fredholm integral equation in the paper of Eringen (1979) is divergent. The kernel of the second kind Fredholm integral equation in Eringen's (1979) paper can be written as follows:

$$L(x,u) = (xu)^{1/2}\int_0^\infty tk(\varepsilon' t)J_0(xt)J_0(ut)\,\mathrm{d}t, \quad 0 \le x, u \le 1,$$

where $J_n(x)$ is the Bessel function of order $n$.

$$k(\varepsilon' t) = -\Phi(\varepsilon' t), \quad \lim_{t\to\infty} k(\varepsilon' t) \neq 0 \quad \text{for} \quad \varepsilon' = \frac{a}{2\beta l} \neq 0,$$

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}}\cos(x - \tfrac{1}{4}\pi) \quad \text{for} \quad x \gg 0.$$

The limit of $tk(\varepsilon' t)J_0(xt)J_0(ut)$ is unequal to zero for $t \to \infty$. So the kernel $L(x,u)$ in Eringen's paper is divergent (see, e.g., Eringen, 1979). Of course, the dual integral equations can be considered to be a single integral equation of the first kind with a discontinuous kernel (see, e.g., Eringen, 1977a). It is well-known in the literature that integral equations of the first kind are generally ill-posed in sense of Hadamard, i.e., small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. For overcoming the difficult, the Schmidt method (Morse et al., 1958) is used to solve the dual-integral equations (27)–(30). The displacement $w$ and the electric potential $\phi$ can be represented by the following series:

$$w(x,0) = \sum_{n=1}^{\infty} a_n P_{2n-1}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{1/2}, \quad \text{for} \quad -l \le x \le l,\ y = 0, \tag{31}$$

$$w(x,0) = 0, \quad \text{for} \quad |x| > l,\ y = 0, \tag{32}$$

$$\phi(x,0) = \sum_{n=1}^{\infty} b_n P_{2n-1}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{1/2}, \quad \text{for} \quad -l \le x \le l,\ y = 0, \tag{33}$$

$$\phi(x,0) = 0, \quad \text{for} \quad |x| > l,\ y = 0, \tag{34}$$

where $a_n$ and $b_n$ are unknown coefficients to be determined and $P_n^{(1/2,1/2)}(x)$ is a Jacobi polynomial (Gradshteyn and Ryzhik, 1980). The Fourier transformation of Equations (31) and (33) is (Erdelyi, 1954)

$$A(s) = \overline{w}(s,0) = \sum_{n=1}^{\infty} a_n B_n \frac{1}{s} J_{2n-1}(sl), \tag{35}$$

$$B(s) = \overline{\phi}(s,0) - \frac{e_{15}}{\varepsilon_{11}}\overline{w}(s,0) = \sum_{n=1}^{\infty}\left(b_n - \frac{e_{15}}{\varepsilon_{11}}a_n\right)B_n\frac{1}{s}J_{2n-1}(sl), \tag{36}$$

$$B_n = 2\sqrt{\pi}(-1)^{n-1}\frac{\Lambda(2n-\frac{1}{2})}{(2n-2)!}, \tag{37}$$

where $\Lambda(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substituting Equations (35) and (36) into Equations (27)–(30), respectively, Equations (28) and (30) can be automatically satisfied, respectively. Then the remaining equations (27) and (29) reduce to the form, respectively.

$$\sum_{n=1}^{\infty} a_n B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(sl)\cos(sx)\, \mathrm{d}s = -\frac{\pi}{2\mu}\tau_0(1+\lambda), \tag{38}$$

$$\sum_{n=1}^{\infty}\left(b_n - \frac{e_{15}}{\varepsilon_{11}}a_n\right)B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(sl)\cos(sx)\, \mathrm{d}s = \frac{\pi D_0}{2\varepsilon_{11}}, \tag{39}$$

where

$$\lambda = \frac{e_{15}D_0}{\varepsilon_{11}\tau_0}.$$

From Equations (38) and (39), it can be shown that the unknown coefficients $a_n$ and $b_n$ have relation as following:

$$b_n = \left(\frac{e_{15}}{\varepsilon_{11}} - \frac{D_0\mu}{\varepsilon_{11}T_0}\right)a_n, \quad T_0 = \tau_0(1+\lambda).$$

For a large $s$, the integrands of Equations (38) and (39) almost decrease exponentially. So the semi-infinite integral in Equations (38) and (39) can be evaluated numerically by Filon's method (see, e.g., Amemiya et al., 1969). Equations (38) and (39) can now be solved for the coefficients $a_n$ and $b_n$ by the Schmidt method (Morse et al., 1958). For brevity, Equation (38) can be rewritten as

$$\sum_{n=1}^{\infty} a_n E_n(x) = U(x),_l < x < l, \tag{40}$$

where $E_n(x)$ and $U(x)$ are known functions and coefficients $a_n$ are unknown and will be determined. A set of functions $P_n(x)$ which satisfy the orthogonality condition

$$\int_{-l}^{l} P_m(x)P_n(x)\,\mathrm{d}x = N_n\delta_{mn}, \quad N_n = \int_{-l}^{l} P_n^2(x)\,\mathrm{d}x \tag{41}$$

can be constructed from the function, $E_n(x)$, such that

$$P_n(x) = \sum_{i=1}^{n}\frac{M_{in}}{M_{nn}}E_i(x), \tag{42}$$

where $M_{ij}$ is the cofactor of the element $d_{ij}$ of $D_n$, which is defined as

*Table 1.* Values of $a_n/(\pi\tau_0/2\mu)$ for $a/2\beta l = 0.0005$

| $n$ | $a_n/(\pi\tau_0/2\mu)$ |
|---|---|
| 1 | −0.318698E+00 |
| 2 | −0.127109E−01 |
| 3 | 0.708155E−02 |
| 4 | 0.164376E−02 |
| 5 | 0.127016E−02 |
| 6 | −0.132851E−03 |
| 7 | −0.570583E−04 |
| 8 | −0.981545E−04 |
| 9 | −0.106541E−04 |
| 10 | −0.582841E−05 |

$$D_n = \begin{bmatrix} d_{11}, & d_{12}, & d_{13}, & \ldots, & d_{1n} \\ d_{21}, & d_{22}, & d_{23}, & \ldots, & d_{2n} \\ d_{31}, & d_{32}, & d_{33}, & \ldots, & d_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d_{n1}, & d_{n2}, & d_{n3}, & \ldots, & d_{nn} \end{bmatrix}, \quad d_{ij} = \int_{-l}^{l} E_i(x)E_j(x)\,\mathrm{d}x. \tag{43}$$

Using Equations (40)–(43), we obtain

$$a_n = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \tag{44}$$

with

$$q_j = \frac{1}{N_j}\int_{-l}^{l} U(x)P_j(x)\,\mathrm{d}x. \tag{45}$$

## 5. Numerical calculations and discussion

Coefficients $a_n$ and $b_n$ can be obtained by solving Equation (40) and the relation $b_n = (e_{15}/\varepsilon_{11} - D_0\mu/\varepsilon_{11}T_0)a_n$. For a check of the coefficients $a_n$, the values of the coefficients $a_n$ are given in Table 1 for $a/2\beta l = 0.0005$.

From the references (see, e.g., Itou, 1978, 1979; Zhou, 1998, 1999), it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of infinite series to Equation (40) are retained. The behavior of the stress stays steady with the increasing number of terms in (40). Coefficients $a_n$ and $b_n$ are known, so that entire stress field and the electric displacement can be obtainable. However, in fracture mechanics, it is of importance to determine stress $\tau_{yz}$ and the electric displacement $D_y$ in the vicinity of the crack's tips. $\tau_{yz}$ and $D_y$ along the crack line can be expressed respectively as

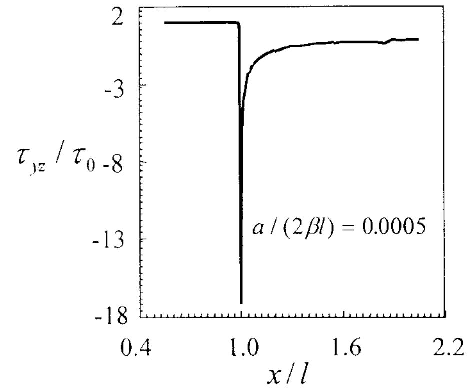


*Figure 2.* Antiplane shear stress.

$$\tau_{yz}(x,0) = -\frac{2}{\pi}\sum_{n=1}^{\infty}(c_{44}a_n + e_{15}b_n)B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(sl)\cos(xs)\,\mathrm{d}s$$

$$= -\frac{2\mu}{\pi(1+\lambda)}\sum_{n=1}^{\infty} a_n B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(sl)\cos(xs)\,\mathrm{d}s, \qquad (46)$$

$$D_y(x,0) = -\frac{2}{\pi}\sum_{n=1}^{\infty}(e_{15}a_n - \varepsilon_{11}b_n)B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(sl)\cos(xs)\,\mathrm{d}s$$

$$= -\frac{2D_0\mu}{\pi T_0}\sum_{n=1}^{\infty} a_n B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(sl)\cos(xs)\,\mathrm{d}s = \frac{D_0}{\tau_0}\tau_{yz}(x,0). \qquad (47)$$

So long as $\varepsilon \neq 0$, the semi-infinite integration and the series in Equations (46) and (47) are convergent for any variable $x$. Equations (46) and (47) give finite stress and electric displacement all along $y = 0$, so there is no stress singularity at the crack tips. However, for $\varepsilon = 0$, we have the classical stress singularity at the crack tips. At $-l < x < l$, $\tau_{yz}/\tau_0$ and $D_y\tau_0/D_0$ are very close to unity, and for $x > l$, $\tau_{yz}/\tau_0$ and $D_y\tau_0/D_0$ possess finite values diminishing from a finite value at $x = l$ to zero at $x = \infty$. Since $\varepsilon/l > 1/100$ represents a crack length of less than 100 atomic distances (as stated by Eringen, 1979), and such submicroscopic sizes other serious questions arise regarding the interatomic arrangements and force laws, we do not pursue solutions valid at such small crack sizes. The semi-infinite numerical integrals, which occur, are evaluated easily by Filon's method (see, e.g., Amemiya et al., 1969) and Simpson's methods because of the rapid diminution of the integrands. In all computation, the material constants are not considered because of the stress field does not depend on the material constants. The lattice parameter is just considered in this paper. Because the integrands of Equations (46) and (47) are complex, the stress along the crack face has a slight variation. Here, we just give the stress field in this paper. The electric displacement field can be obtained by the stress field using Equation (47). The results are plotted in Figures 2–7.

The following observations are very significant:

(i) For $\varepsilon \neq 0$, it can be proved that the semi-infinite integration

$$\int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(sl)\cos(xs)\,\mathrm{d}s$$

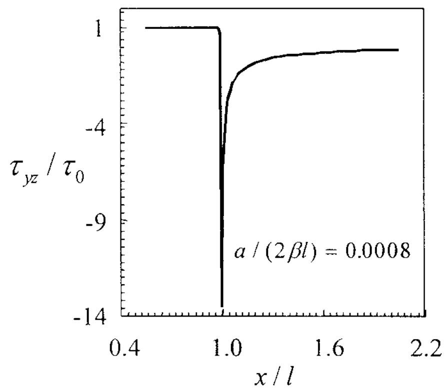


*Figure 3.* Antiplane shear stress.

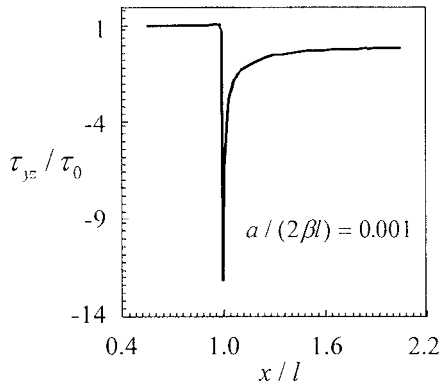


*Figure 4.* Antiplane shear stress.

and the series

$$\sum_{n=1}^{\infty} a_n B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(sl) \cos(xs)\, \mathrm{d}s$$

in Equations (46)–(47) are convergent for any variable $x$. So the stress and the electric displacement give finite values all along the crack line. Contrary to the classical piezoelectric

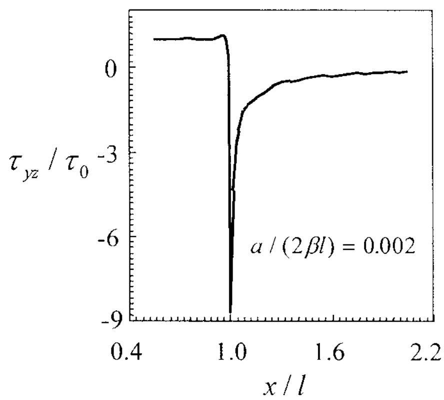


*Figure 5.* Antiplane shear stress.

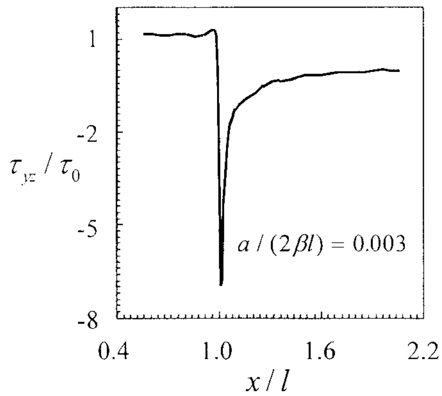


*Figure 6.* Antiplane shear stress.

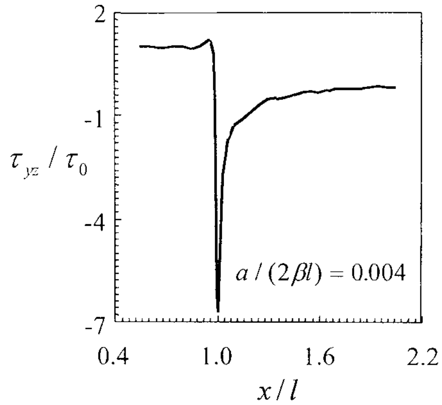


*Figure 7.* Antiplane shear stress.

theory solution, it is found that no stress and electric displacement singularity are present at the crack tip, and also the present results converge to the classical ones when far away from the crack tip. The maximum stress does not occur at the crack tip, but slightly away from it. This phenomenon has been thoroughly substantiated by Eringen (Eringen, 1983). The distance between the crack tip and the maximum stress point is very small and it depends on the crack length and the lattice parameter.

(ii) The stress at the crack tip becomes infinite as the atomic distance $a \to 0$. This is the classical continuum limit of square root singularity. This can be shown from Equations (27)–(30). For $a \to 0$, $\mathrm{erfc}(\varepsilon s) = 1$, Equations (27)–(30) will reduce to the dual integral equations for the same problem in classical piezoelectric materials. These dual integral equations can be solved by using the singular integral equation for the same problem in the local piezoelectric materials problem. However, the stress and the electric displacement singularity are present at the crack tip in the local piezoelectric materials problem as well known.

(iii) For the $a/\beta$ = constant, viz., the atomic distance does not change, the value of the stress concentrations (at the crack tip) becomes higher with the increase of the crack length ($a/2\beta l$ will become smaller with the increase of the crack length $l$). Note this fact, experiments indicate that the piezoelectric materials with smaller cracks are more resistant to fracture than those with larger cracks.

(iv) The significance of this result is that the fracture criteria are unified at both the macroscopic and microscopic scales, viz., it may solve the problem of any scale cracks (it may solve the problem of any value of $a/2\beta l$).

(v) The present results will revert to the classical ones when the introduction function $a(|X' - X|) = \delta(|X' - X|)$.

(vi) The stress concentration occurs at the crack tip as stated by Eringen (Eringen, 1978, 1979), and this is given by

$$\tau_{yz}(l, 0)/\tau_0 = c_3/\sqrt{a/2\beta l}, \tag{48}$$

where $c_3$ converges to $c_3 = 0.383$.

(vii) The dimensionless stress field is found to be independent of the electric loads and the material parameters. They just depend on the length of the crack and the lattice parameter. However, the electric displacement is found to depend on the stress loads, the length of the crack and the lattice parameter. The stress field is not coupled with the electric field. This is consistent with the piezoelectric theory.

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